UNIVERSITÄT ZU LÜBECK

Finding minimal *d*-separators in linear time and applications

Benito van der Zander, Maciej Liśkiewicz

24th July, UAI 2019, Tel Aviv

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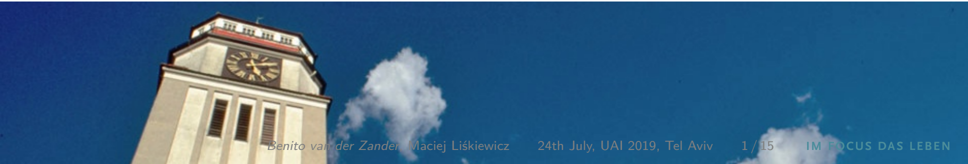

Motivation and our Contribution

We investigate algorithmic aspects of the following problem:

Problem

For given nodes X, Y in a causal DAG, find a minimal d-separator between X and Y.

- Minimal d-separators are an important graphical notion used in causality
- Many problems can be reduced to finding (minimal) d-separators
- The best previously known algorithm for minimal separators needs $\mathcal{O}(n^2)$ time [Tian, Paz, and Pearl 1998]

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- Many problems can be reduced to finding (minimal) d-separators
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Our main result:

Finding minimal d-separators can be done in linear time

Our result also holds for classes beyond DAGs and minimal d-separators between node sets $\mathbf{X}, \mathbf{Y}$.

Preliminaries: *d*-separation in causal DAGs

Graph reveals conditional independences

- A path is a sequence of adjacent nodes
- A node V on a path is a *collider* if the adjacent edges are $\rightarrow V \leftarrow$
- A path is *blocked* if it contains a collider
  (i.e., only edges $\rightarrow V \rightarrow$, $\leftarrow V \leftarrow$, and $\leftarrow V \rightarrow$ allowed)
- Two nodes are *d-separated* if all paths between them are blocked
- Two nodes are *d-connected* if they are not *d*-separated
  (i.e., there exist any unblocked path between them)

X and Y are correlated iff X and Y are *d*-connected

X and Y are independent iff X and Y are *d*-separated

Preliminaries: *d*-separation in causal DAGs

Conditioning on a set **Z** inverts the rules

	$V \notin \mathbf{Z}$	$V \in \mathbf{Z}$
$\rightarrow V \leftarrow$	blocked (unless a descendant is in **Z**)	allowed
$\rightarrow V \rightarrow$	allowed	blocked
$\leftarrow V \rightarrow$	allowed	blocked
$\leftarrow V \leftarrow$	allowed	blocked

A path is blocked, if any node on it is blocked

X and Y are conditionally independent given **Z** iff all paths between X and Y are blocked given **Z**

Minimal *d*-separators

Definition

A given set $\mathbf{Z}$ is a *minimal d-separator* between X and Y if

- $\mathbf{Z}$ d-separates X and Y
- No $\mathbf{Z}' \subsetneq \mathbf{Z}$ d-separates X and Y

Minimal d-separators

Fact ([Tian, Paz, and Pearl, 1998])

A given set $\mathbf{Z}$ *is a minimal d-separator between* X *and* Y *if*

- $\mathbf{Z}$ *d-separates* X *and* Y
- *For each* $Z \in \mathbf{Z}$, *the set* $\mathbf{Z} \setminus Z$ *does not d-separate* X *and* Y

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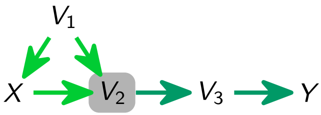
Example:



Minimal d-separator: $\mathbf{Z} = \{V_2\}$

Minimal d-separators

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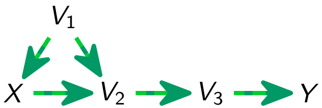

Minimal d-separator: $\mathbf{Z} = \{V_2\}$
Not a (minimal) d-separator: $\mathbf{Z} = \{\}$

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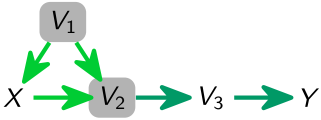

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Not a minimal d-separator: $\mathbf{Z} = \{V_1, V_2\}$

Minimal d-separators

Additional useful constraint $\mathbf{I} \subseteq \mathbf{Z} \subseteq \mathbf{R}$ for some $\mathbf{I} \subseteq \mathbf{R}$

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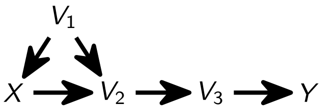

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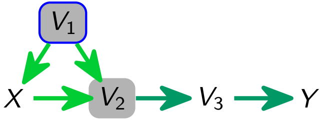


Minimal d-separator: $\mathbf{Z} = \{V_2\}$
Not a (minimal) d-separator: $\mathbf{Z} = \{\}$
Not a minimal d-separator: $\mathbf{Z} = \{V_1, V_2\}$ under the constraint $\mathbf{I} = \{\}$
Minimal d-separator: $\mathbf{Z} = \{V_1, V_2\}$ under the constraint $\mathbf{I} = \{V_1\}$

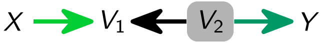
Minimal *d*-separators: Collider examples

$$X \rightarrow V_1 \leftarrow V_2 \rightarrow Y$$

Minimal *d*-separator: $\mathbf{Z} = \emptyset$

Minimal d-separators: Collider examples

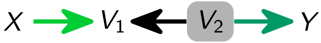
Minimal d-separator: $\mathbf{Z} = \emptyset$



Not a minimal d-separator $\mathbf{Z} = \{V_2\}$

Minimal d-separators: Collider examples

Minimal d-separator: $\mathbf{Z} = \emptyset$



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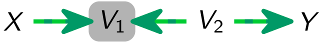

Not a (minimal) d-separator: $\mathbf{Z} = \{V_1\}$

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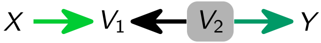

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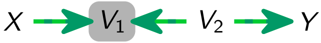

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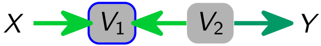

Not a minimal d-separator: $\mathbf{Z} = \{V_1, V_2\}$ under constraint $\mathbf{I} = \{\}$
A minimal d-separator: $\mathbf{Z} = \{V_1, V_2\}$ under constraint $\mathbf{I} = \{V_1\}$

Minimal d-separators in causal DAGs

$An(X, Y, \mathbf{I})$ = ancestors of X, Y and $\mathbf{I}$

Fact

A given set $\mathbf{Z}$ is a minimal d-separator between X and Y if

- *Every path in $An(X, Y, \mathbf{I})$ between X and Y intersects $\mathbf{Z}$ in a non-collider*
- *For each $Z \in \mathbf{Z} \setminus \mathbf{I}$ there are paths*

  π_{XZ} in $An(X, Y, \mathbf{I})$ from X to Z and
  π_{YZ} in $An(X, Y, \mathbf{I})$ from Y to Z

  which contain no non-collider node of $\mathbf{Z} \setminus Z$

Main Results

Let X^* be the nodes reachable from X through $An(X, Y, \mathbf{I})$ by paths containing no non-collider in $\mathbf{Z}$
Let Y^* be the nodes reachable from Y analogously

Fact (Minimality Criterion)

A given set $\mathbf{Z}$ is a minimal d-separator between X and Y if

- $Y \notin X^*$ *(equivalently: $X \notin Y^*$)*
- $\mathbf{Z} \setminus \mathbf{I} \subseteq X^* \cap Y^*$

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Linear time algorithm to test if a given $\mathbf{Z}$ is a minimal d-separator:

- Calculate X^* and Y^*
- Verify conditions of the minimality criterion

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Linear time algorithm to find a minimal d-separator $\mathbf{Z}$:

- Start with non-minimal d-separator $\mathbf{Z}_0 := An(X, Y, \mathbf{I}) \cap \mathbf{R}$
- Calculate X^* for $\mathbf{Z}_0$
- Remove unnecessary nodes $\mathbf{Z}_X := \mathbf{Z}_0 \cap X^* \cup \mathbf{I}$
- Calculate Y^* for $\mathbf{Z}_X$
- Remove further unnecessary nodes and return $\mathbf{Z} = \mathbf{Z}_X \cap Y^* \cup \mathbf{I}$

Applications: Adjustment sets

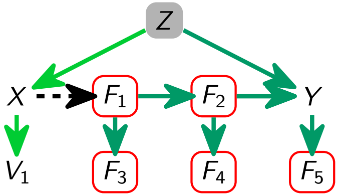
Definition

A set $\mathbf{Z}$ is an adjustment set relative to X and Y if the causal effect of X on Y is given by

$$P(Y \mid do(X)) = \sum_{\mathbf{z}} P(Y \mid X, \mathbf{Z}) P(\mathbf{Z})$$

Theorem (van der Zander et al., UAI 2014)

A set $\mathbf{Z}$ is an adjustment set relative to X and Y iff it contains no node of $\mathrm{Forbidden}(X, Y)$ and is a d-separator in the proper backdoor graph $G^{pbd}_{\mathbf{XY}}$



Forbidden: $\{F_1, F_2, F_3, F_4, F_5\}$

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Proposition

Using our algorithms we can construct minimal adjustment sets in linear time

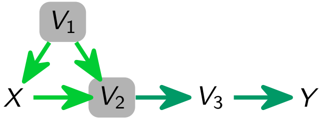
Applications: Nearest Separator

Definition

A set $\mathbf{Z} \subseteq \mathbf{R} \setminus \{X, Y\}$ is called a *nearest separator* relative to X and Y if

(a) $\mathbf{Z}$ d-separates X and Y, and

(b) for any $Z \in \mathbf{Z}$ and any set $\mathbf{W} \subseteq \mathbf{R} \setminus \{X, Y, Z\}$ that d-separates X and Y, it holds:
$\mathbf{W}$ d-separates Z and Y



Nearest separators: $\mathbf{W} = \{V_2\}$ and $\mathbf{W} = \{V_1, V_2\}$

Not a nearest separator: $\mathbf{W}' = \{V_3\}$

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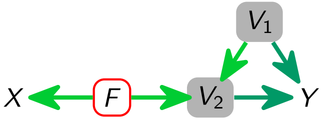


Forbidden node F, i.e., constraint $\mathbf{R} = \{V_1, V_2\}$

Nearest separator $\mathbf{W} = \{V_1, V_2\}$

Applications: Nearest Separator

Definition

A set $\mathbf{Z} \subseteq \mathbf{R} \setminus \{X, Y\}$ is called a *nearest separator* relative to X and Y if

(a) $\mathbf{Z}$ *d*-separates X and Y, and

(b) for any $Z \in \mathbf{Z}$ and any set $\mathbf{W} \subseteq \mathbf{R} \setminus \{X, Y, Z\}$ that *d*-separates X and Y, it holds:
$\mathbf{W}$ *d*-separates Z and Y

Our algorithm always returns *d*-separators that are minimal and nearest separators

Proposition

In a linear graphical model (SEM) nearest separators ...

... *used as adjustment sets have the optimal asymptotic variance among all adjustment sets that are subsets of* $An(Y, X, \mathbf{I})$

... *allow to find conditional instrumental variables (next slides)*

Applications: Conditional Instrumental Variables

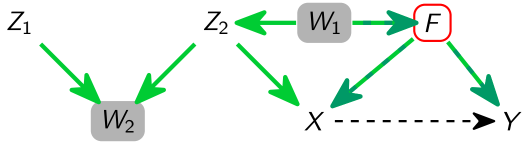
Definition (Conditional instrumental variable)

Variable Z is a *conditional instrument* relative to edge $X \to Y$, if there exists a d-separator $\mathbf{W} \subseteq \mathbf{R}$ such that

(a) $\mathbf{W}$ does not d-separate Z and X,

(b) $\mathbf{W}$ d-separates Z and Y after removal of the edge $X \to Y$,

(c) $\mathbf{W}$ consists of non-descendants of Y



Conditional instrument: $Z = Z_1$, $\mathbf{W} = \{W_1, W_2\}$
Conditional instrument with nearest separator: $Z = Z_2$, $\mathbf{W} = \{W_1\}$

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Given a conditional instrument Z the direct causal effect of $X \to Y$ is

$$\frac{\mathrm{Cov}(Y, Z|\mathbf{W})}{\mathrm{Cov}(X, Z|\mathbf{W})}$$

Applications: Conditional Instrumental Variables

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Theorem (van der Zander et al., IJCAI 2015)

- *Testing whether a given variable Z is a conditional instrumental variable is NP-complete, because finding the d-separator $\mathbf{W}$ is NP-complete*
- *If any $\mathbf{W} \subseteq An(Y, X)$ satisfies the definition, any nearest separator between Y and Z can be used as d-separator $\mathbf{W}$*
- *If any conditional instrumental variable exists for given X, Y, then there exists one with its $\mathbf{W} \subseteq An(Y, X)$*

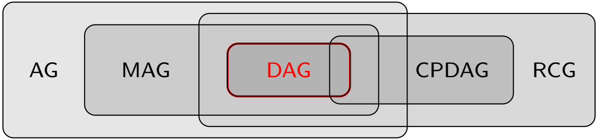
Minimal separators in DAGs: Summary

We have improved the runtime of the following problems:

	Previously	Our Results
Verify a minimal d-separator	$\mathcal{O}(n^2)$ [Tian, Paz, and Pearl, 1998]	$\mathcal{O}(n+m)$
Find a minimal d-separator	$\mathcal{O}(n^2)$ [Tian, Paz, and Pearl, 1998]	$\mathcal{O}(n+m)$
Verify a minimal adjustment set	$\mathcal{O}(n^2)$ [van der Zander, Textor, and Liśkiewicz, UAI 2014]	$\mathcal{O}(n+m)$
Find a minimal adjustment set	$\mathcal{O}(n^2)$ [van der Zander, Textor, and Liśkiewicz, UAI 2014]	$\mathcal{O}(n+m)$
Prune an adjustment set	$\mathcal{O}(n(n+m))$ [Henckel, Perković, and Maathuis, 2019]	$\mathcal{O}(n+m)$
Find a nearest d-separator	$\mathcal{O}(n(n+m))$ [van der Zander, Textor, and Liśkiewicz, IJCAI 2015]	$\mathcal{O}(n+m)$
Find a conditional instrument	$\mathcal{O}(n^2(n+m))$ [van der Zander, Textor, and Liśkiewicz, IJCAI 2015]	$\mathcal{O}(n(n+m))$

Causal Models beyond DAGs

Our algorithms for minimal separators work for



Directed acyclic graphs (DAGs) the classic model

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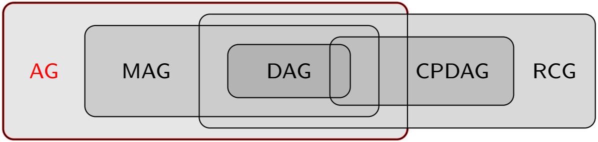

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Ancestral Graphs (AGs) with latent variables and selection bias

Causal Models beyond DAGs

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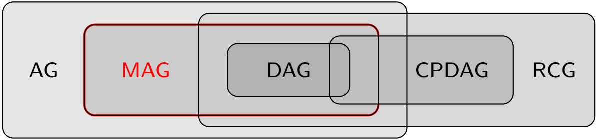

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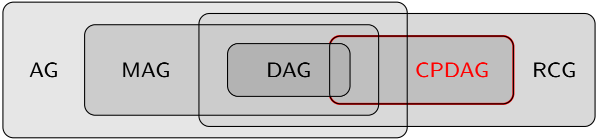

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Completed Partially Directed Acyclic Graph (CPDAGs)
represent entire class of Markov equivalent DAGs

Causal Models beyond DAGs

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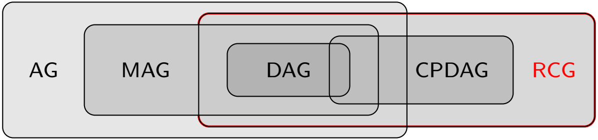

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Restricted Chain Graphs (RCGs) generalization of CPDAGs, represent a set of Markov equivalent DAGs

Causal Models beyond DAGs

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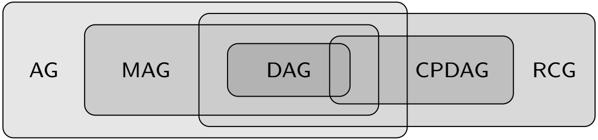

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Thank you for your attention