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THEORETISCHE INFORMATIK

Separators and Adjustment Sets in Markov Equivalent DAGs

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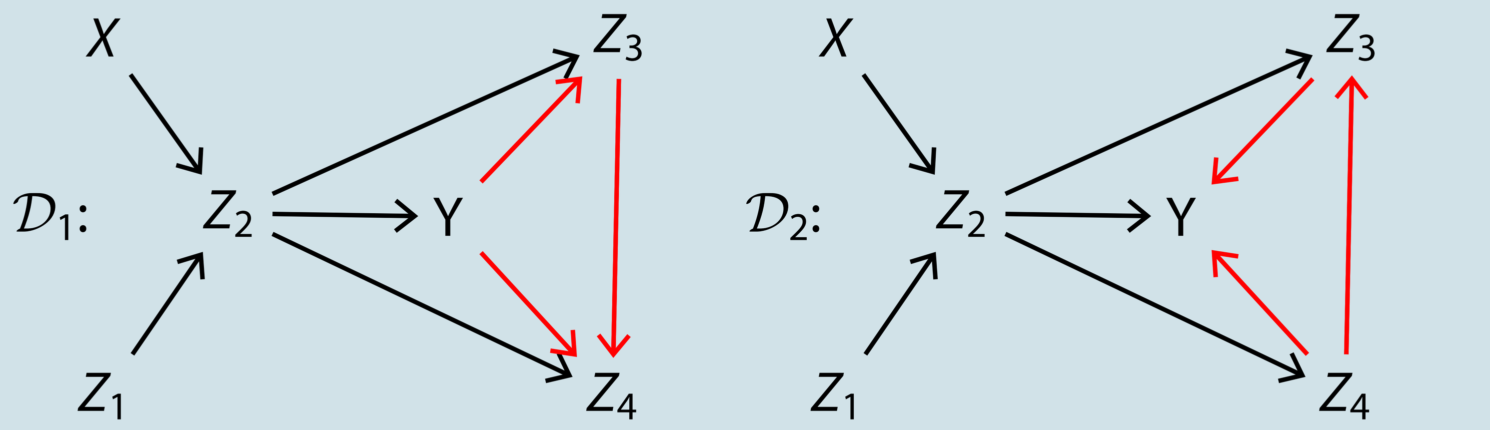
1. Motivation

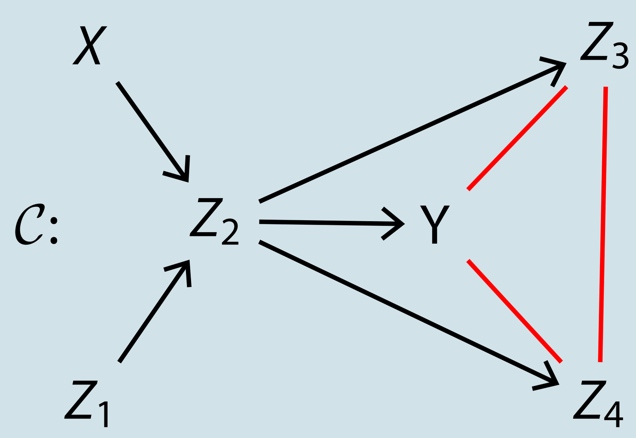
DAGs (directed acyclic graphs)[1]
The causal relationships between variables can be encoded as DAG:



Such a DAG implies a probability distribution P that can be factorized on nodes given their parents to

$$P(\mathbf{v}) = \prod_{j=1}^{n} P(x_j|pa_j)$$

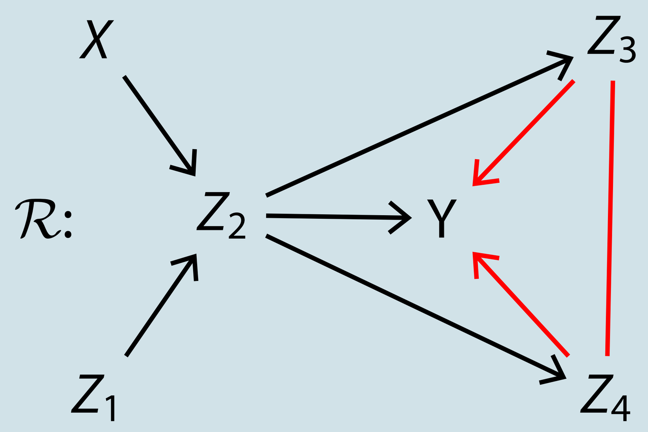
Only the distribution itself is learnable from observed data, so learning algorithms return multiple DAGs that encode the same distribution and form an equivalence class $[\mathcal{D}_i]$. They correspond to **Complete Partial DAGs (CPDAGs)**[2]



CPDAG: a mixed graph $\mathcal{C}$ representing an entire equivalence class $CE(\mathcal{C}) = [\mathcal{D}_i]$. All $\mathcal{D}_i$ have the same directed edges as $\mathcal{C}$ and an directed edge for every undirected edge.

Chain Graphs (CGs)[3] CGs generalize CPDAGs and are mixed graphs that do not contain a semicycle. CGs that are not complete encode a subset $CE(\mathcal{G}) \subset [\mathcal{D}_i]$, which might be empty as $CE(\mathcal{G}) = \emptyset$.

Restricted Chain Graphs (RCGs)
are chordal and do not contain $A \to B - C$. These conditions ensure that $CE(\mathcal{R}) \neq \emptyset$ for every RCG $\mathcal{R}$. Every CPDAG and every DAG is an RCG. And for every CG $\mathcal{G}$ with $CE(\mathcal{G}) \neq \emptyset$, we can find an RCG $\mathcal{R}$ with $CE(\mathcal{R}) = CE(\mathcal{G})$ efficiently.



2. Covariate Adjustment

Given a causal DAG the causal effect of a perfect experiment that sets the variables $\mathbf{X}$ to $\mathbf{x}$ can be calculated with the *do-operator* as

$$P(\mathbf{y}|do(\mathbf{x})) = \prod_{X_j \in \mathbf{V} \setminus \mathbf{X}} P(x_j|pa_j).$$

For certain sets $\mathbf{Z}$ this causal effect can also be calculated as

$$P(\mathbf{y}|do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{y}|\mathbf{x},\mathbf{z})P(\mathbf{z})$$

directly from the data and mostly independently of the DAG itself. These sets are called *adjustment* [1].
From a DAG it can be decided, if a given set is an adjustment set by:

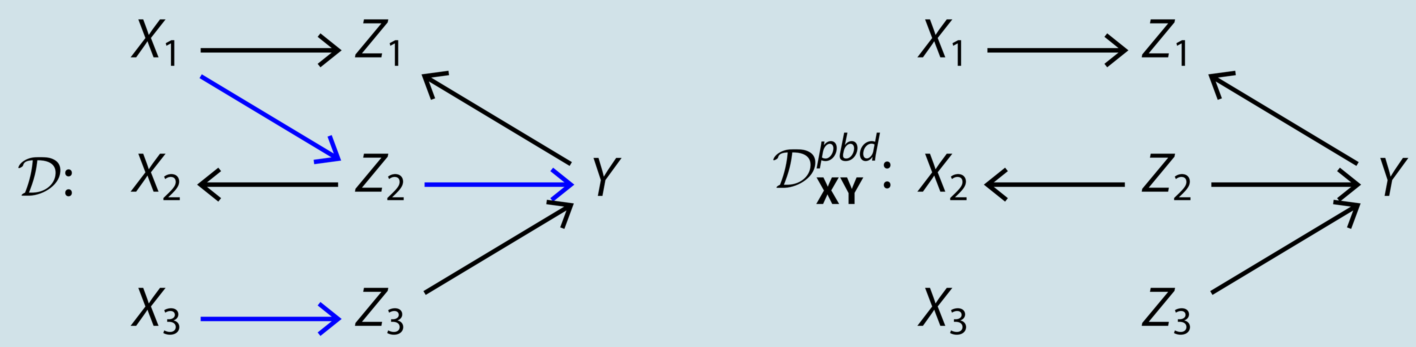
Adjustment Criterion (AC) for DAGs [4]: Let $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ be a DAG and let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be pairwise disjoint subsets of $\mathbf{V}$. The set $\mathbf{Z}$ satisfies the *adjustment criterion* relative to $(\mathbf{X}, \mathbf{Y})$ if

1. no element in $\mathbf{Z}$ is a descendant of any $W \in \mathbf{V} \setminus \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$ and
2. all proper non-causal paths from $\mathbf{X}$ to $\mathbf{Y}$ are blocked by $\mathbf{Z}$.

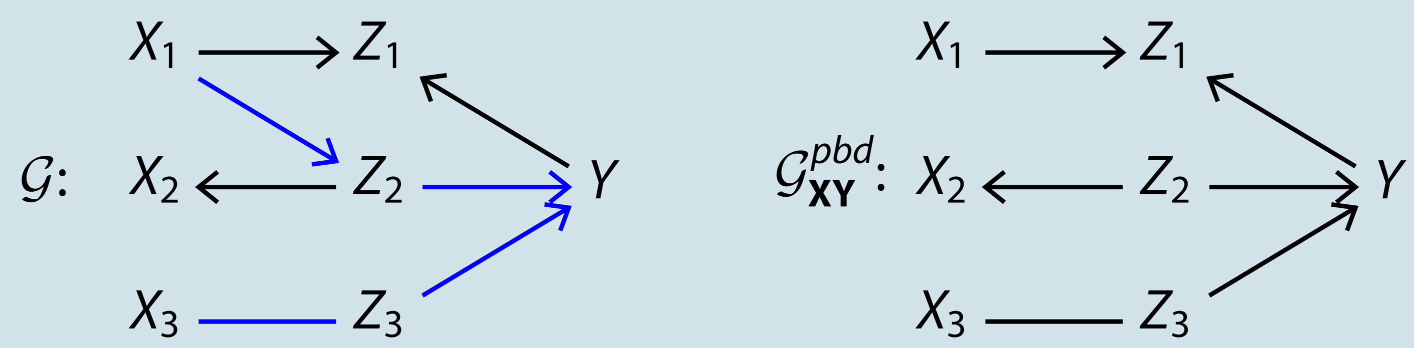
For a distribution described by a CG $\mathcal{G}$ a set $\mathbf{Z}$ can only be an adjustment, if it is an adjustment in every represented DAG $\mathcal{D} \in CE(\mathcal{G})$. Testing the above criterion for each of these DAGs is not feasible, so we need a new criterion that can be tested directly on $\mathcal{G}$.

3. Constructive Back-Door Criterion for (R)CGs

The second condition of the AC depends on "all non-causal paths." Even in DAGs there can be an exponential number of these paths, so the AC does not lead directly to efficient algorithms. However, after removing the first directed edge of every directed path, this condition becomes equivalent to d-separation in a new graph:



We show that this reduction can be generalized to RCGs:



We call this graph a proper back-door graph and obtain the following simpler criterion for RCGs:
Constructive Back-Door Criterion (CBC): Let $\mathcal{R} = (\mathbf{V}, \mathbf{E})$ be an RCG and let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be pairwise disjoint subsets of $\mathbf{V}$. The set $\mathbf{Z}$ satisfies the *CBC* relative to $(\mathbf{X}, \mathbf{Y})$ if

1. no element in $\mathbf{Z}$ is a possible descendant of any $W \in \mathbf{V} \setminus \mathbf{X}$ which lies on a proper possible causal path from $\mathbf{X}$ to $\mathbf{Y}$ and
2. all definite status paths in the proper back-door graph are blocked by $\mathbf{Z}$.

If an adjustment set exists, the proper back-door graph is also an RCG.

4. Algorithms

The d-connected paths of DAGs generalize to RCGs by also allowing passing through $A \leftarrow B - C$, $A - B \to C$, and $A - B - C$.
This leads to algorithms for these problems in an RCG (DAG, CPDAG):

Testing or finding an sep. or adj. set $\mathbf{Z}$	in linear time $O(n+m)$
Testing or finding a minimal set $\mathbf{Z}$	$O(n^2)$ using a moral graph
Testing or finding a minimum set $\mathbf{Z}$	$O(n^3)$ using max-flow
Enumerating all set $\mathbf{Z}, \mathbf{Z}', \mathbf{Z}'', \ldots$	delayed $O(nm)$
Enumerating all minimal set $\mathbf{Z} \ldots$	delayed $O(n^3)$

satisfying the constraint $\mathbf{I} \subseteq \mathbf{Z} \subseteq \mathbf{R}$ for any given node sets $\mathbf{I}, \mathbf{R}$.
n and m denote the number of nodes and edges in the graph.

In an arbitrary chain graph $\mathcal{G}$ we can solve these problems after converting it to an RCG $\mathcal{R}$ in $O(n^4)$ by replacing every occurrence of $A \to B - C$ with $A \to B \to C$. If this replacement is not unique determined, i.e. if there is a D with $B - C \leftarrow D$, then $CE(\mathcal{G}) = \emptyset$.

5. Conclusion

Problems concerning adjustment sets in chain graphs can be reduced to d-separation problems in RCGs, a new class including DAGs and CPDAGs. These problems can be solved by efficient and easily implementable algorithms [http://www.dagitty.net].

6. References

[1] Judea Pearl. *Causality*. Cambridge University Press, 2009.
[2] Steen A Andersson, David Madigan, and Michael D Perlman. 1997.
[3] Steffen Lauritzen and Nanny Wermuth. 1989.
[4] Ilya Shpitser, Tyler VanderWeele, and James Robins. 2010.